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ABSTRACT

A method is presented for the analytic design of linear control systems. Design is accomplished by the solution of an algebraic set of simultaneous equations derived from the design specifications. Specifications considered include the system damping ratio, undamped natural frequency, bandwidth, steady state error coefficients and root sensitivity.

The method is shown to be fast, accurate, specific and capable of working with any number of parameters. In addition, the analytic approach minimizes dependence on graphical techniques and trial and error solutions.

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SYMBOLS AND ABBREVIATIONS

CE	Characteristic equation
E_{ss}	Steady state error
G(s)	Forward transfer function
k_a	Acceleration error constant
K_a	Accelerometer gain
K	Gain
K_p or k_p	Position error constant
K_v or k_v	Velocity error constant
K_t	Tachometer gain
M_{pw}	Reasonance peak
Superscript or subscript "o"	Specified or numerical value
w (omega)	Angular frequency
w_n	Undamped natural frequency
w_b	Bandwidth (-3 db)
w_r	Reasonant frequency
$\phi(\zeta)$	Polynomial in ζ
Q	General sensitivity parameter
s	Complex variable of the form $\sigma + jw$
S	Sensitivity
S^P_Q	Sensitivity of P with respect to Q

S_Q^{ζ}	ζ component of root sensitivity
$S_Q^{w_n}$	w_n component of root sensitivity
S_Q^{RT}	Total root sensitivity
ζ (zeta)	Damping ratio

1. Introduction.

Linear control system design can be defined as "a set of procedures for determining the value of system parameters which will cause a physical system to operate in a prescribed manner." The set of procedures noted above is as broad as the definition. One may choose to perform the design effort in the time domain by applying modern state space concepts or by direct solution of the system differential equations. Alternatively, the complex frequency domain may be selected, employing any of the universally accepted procedures such as Nichols and Bode plots, root locus, Mitrovic's Method, etc.

Notwithstanding the achievements which have resulted from direct application of the above procedures, it must be admitted that they have inherent limitations. With the exception of simple systems, time domain analysis virtually demands the assistance of a digital computer. In a sense, this is a disadvantage because computers are neither available to all people at all times nor is it economically practical to employ a computer for all computation. The complex frequency schemes are considered imperfect because they are time consuming, i.e., essentially graphical in nature and often based on trial and error, and normally limited to one or two variables.

In view of the above discussion, it appears that there is need for still another design tool. However, in order for a new procedure to even merit consideration, it must be relatively free from the faults and limitations cited above. Specifically, the method should be analytically simple, i.e. extend the range of problems which can be solved without resorting to graphical techniques or the exclusive use of a computer. It should also be rapid, capable of handling several variables, specific and accurate. The remainder of this paper is devoted to the development and discussion of a design procedure which is believed to have these attributes.

2. The Algebraic Method: Concept and Development

Concept

The proposed design procedure is based on the premise that system design specifications can be transformed into algebraic functions of the system parameters. If the specifications are independent, N specifications will produce N equations which simultaneously constrain the system. Control system design would thus revert to the relatively simple problem of solving a set of simultaneous algebraic equations.

The problem of transforming specifications into equations is not as broad and difficult as one might initially assume. The most common and practical complex frequency specifications are: undamped natural frequency (w_n), daming ratio (ζ), steady state error (E_{ss}), bandwidth (w_b), reasonance peak (M_{pw}) and reasonant frequency (w_r). In many cases, both ζ and w_n are specified. This is equivalent to specifying the location of a complex pole pair and generally implies that these poles should dominate the system response. If attention is restricted to this situation, then the specifications of primary interest are ,
w_n, w_b and E_{ss} because M_{pw} and w_r can be at least approximately expressed in terms of ζ and w_n. The remainder of this section is therefore devoted to the development of methods

which can be used to obtain system constraint equations from specifications for ζ , w_n, w_b and E_{ss}.

Root Location (ζ and w_n)

The coefficient equations associated with the specification of a complex pole pair can be developed in two ways. The first method is based on the root-coefficient relations of an algebraic equation; the second is based on Mitrovic's Method.

Assume the systems' characteristic equation (CE), obtained by setting the denominator of the closed loop transfer function equal to zero, is the polynomial.

$$\sum_{K=0}^{n} a_K s_K = 0 \tag{2-1}$$

where s is a complex variable and defined as

$$s \equiv -\zeta \omega_n + j\omega_n\sqrt{1-\zeta^2} \tag{2-2}$$

and the a_k's represent the real coefficients which are functions of the system parameters.

Since equation (2-1) is of order n, there must be n roots which satisfy the equation. Therefore, an equivalent expression for equation (2-1) is

$$\prod_{i=1}^{n} (s + R_i) = 0 \tag{2-3}$$

where R_i represents the ith root of the polynomial. By expanding equation (2-3) and equating the coefficients of like powers of s in equations (2-1) and (2-3), n equations can be obtained which relate the roots and coefficient of the original polynomial.

If ζ_o and w_o define the location of the specified complex pole pair, it is permissible to set

$$R_1 = \zeta_o \omega_o + j\,\omega_o\sqrt{1-\zeta_o^2} \tag{2-4}$$

$$R_2 = \zeta_o \omega_o - j\,\omega_o\sqrt{1-\zeta_o^2} \tag{2-5}$$

Substituting these values of R_1 and R_2 into the n root-coefficient relations obtained previously gives n equations in terms of the remaining n-2 roots and the coefficients of equation (2-1). By algebraic manipulation, it is then possible to successively eliminate each of the remaining n-2 roots and finally obtain two equations involving only the coefficients. Since the coefficients are explicit functions of the system parameters, any combination of the parameters which satisfy these equations will generate the desired pole pair.

Example 2.1

Let the system characteristic equation be

$$s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

Find the relations which must exist between the coefficients in order to locate a pole pair with $\zeta_o = 1/2$ and $w_o = 10$.

Denote the three roots of the CE as R_1, R_2 and R_3. Then

$$(s+R_1)(s+R_2)(s+R_3) = s^3 + a_2 s^2 + a_1 s + a_0$$

and $$a_1 = R_1 R_2 + R_3 (R_1 + R_2)$$

$$a_0 = R_1 R_2 R_3$$

$$a_2 = R_1 + R_2 + R_3$$

Setting $$R_1 = 5 + j10\sqrt{3/4}$$

$$R_2 = 5 - j10\sqrt{3/4}$$

Then $$100(a_2 - 10) = a_0 \qquad (1)$$

$$10(a_1 - 10) = a_0 \qquad (2)$$

$$100 R_3 = a_0 \qquad (3)$$

Equations (1) and (2) are the desired coefficient relations; equation (3) provides information about the third pole.

While the root-coefficient method works well with low ordered systems (and even provides information about the remaining n-2 roots), it requires an inordinate amount of computation when applied to systems of higher order because the coefficient relations must be extracted from a set of non-linear equations. This difficulty can be overcome (at the expense of losing information about the remaining n-2 roots) by using Mitrovic's Method. With this method, a k<u>th</u> ordered system can be handled as easily as one of third order. The method is fully developed in (3) and (5). Only a brief review is given here.

The argument begins with equations (2-1) and (2-2) defined previously. Two independent equations in a_k, w_n and ζ can be obtained by substituting equation (2-2) into equation (2-1) and setting the real and imaginary portions of the resulting expressions equal to zero. Through additional manipulation and the introduction of a new variable, the equations take the form

$$\sum_{K=0}^{n} a_K \, \omega_n^K \, \phi_K(\zeta) = 0 \qquad (2\text{-}6)$$

$$\sum_{K=0}^{n} a_K \, \omega_n^K \, \phi_{K-1}(\zeta) = 0 \qquad (2\text{-}7)$$

where $$\phi_K(\zeta) = -\left[2\zeta\phi_{K-1}(\zeta) + \phi_{K-2}(\zeta)\right] \qquad (2\text{-}8)$$

with $\phi_0(\zeta) = 0$, $\phi_1(\zeta) = -1$ and $\phi_{-K}(\zeta) = -\phi_K(\zeta)$.

Other values of $\phi_k(\zeta)$ for various values of ζ are tabulated in Appendix I.

Equations (2-6) and (2-7) are known as the generalized Mitrovic equations and can be used to obtain the coefficient relations required to locate a pole pair as specified by ζ_o and w_o.

Example 2.2

Work example 2.1 using Mitrovic's Method.

For a third order system, the generalized Mitrovic equations are

$$a_0\phi_0(\zeta) + a_1\omega_n\phi_1(\zeta) + a_2\phi_2(\zeta)\omega_n^2 + a_3\omega_n^3\phi_3(\zeta) = 0$$

$$a_0\phi_{-1}(\zeta) + a_1\omega_n\phi_0(\zeta) + a_2\omega_n^2\phi_1(\zeta) + a_3\omega_n^3\phi_2(\zeta) = 0$$

Substituting $w_n = w_o = 10$ and using values for $\phi_k(1/2)$ from Appendix I, the above equations simplify to

$a_1 = 10$ [illegible] ; $a_o = 100 a_2 - 1000$

The results are equivalent to those obtained previously.

Although only a single pair of complex poles is being considered here, it should be noted that either method is capable of handling any number of real or complex poles (providing, of course, that the total number considered is less or equal than the order of the system). For example, assume two complex poles pairs and two real roots of a sixth order system are specified. The root-coefficient method will provide six constraining equations by direct substitution of the given values. If Mitrovic's Method is used, four constraint equations dependent on the complex poles can be obtained by repeated use of equations (2-6) and (2-7). The remaining two equations which are dependent on the real roots are produced by repeated use of

$$\sum_{K=0}^{n} (-1)^K a_K \sigma_K = 0 \tag{2-8}$$

obtained by letting $s = -\sigma$ in equation (2-1).

Steady State Error (E_{ss})

There are three different steady state error constants which may be specified. They are

position error constant $= k_p = \lim_{s \to 0} G(s)$ (2-9)

velocity error constant $= k_v = \lim_{s \to 0} sG(s)$ (2-10)

acceleration error constant $= k_a = \lim_{s \to 0} s^2G(s)$ (2-11)

The above definitions are only valid for systems which have all their roots in the left-half of the s-plane (because the definitions are merely restatement of the final value theorem for complex variables) and unity feedback around the forward loop transfer function, G(s). To satisfy the last requirement, it may be necessary to manipulate the block diagram into an equivalent G(s) block with the required unity feedback.

The constraint equation is obtained by equating the specified value of E_{ss} with the algebraic expression resulting from application of equation (2-9), (2-10) or (2-11). The procedure is demonstrated in the following examples.

Example 2.3

For the system shown in figure 2-1, obtain an expression between K, Z and P for $k_a = 6$.

For the given system

$$G_{EQ}(s) = \frac{K(s+Z)}{s^2(s+5)(s+6)(s+P)}$$

Applying equation (2-11)

$$k_a = \lim_{s \to 0} s^2 G_{EQ}(s) = \frac{KZ}{30P}$$

which, for $k_a = 6$, reduces to: $\quad 180P = KZ$

Example 2.4

For the system shown in figure 2-2, derive an equation in terms of K_1, K_2 and K_3 which will give $K_v = 8$.

Since the system does not have unity feedback, insert fictitious positive and negative feedback paths around the system. $G_{eq}(s)$ can then be found by including the positive feedback branch.

$$G_{EQ}(s) = \frac{K_1}{s^2(1+K_1K_3)+s(5+K_1K_2)-K_1}$$

Applying equation (2-10)

$$K_V = \lim_{s\to 0} s\, G_{EQ}(s) = 0$$

Therefore the specification cannot be satisfied.

Bandwidth (w_b)

Bandwidth will be defined as the angular frequency, w_b, at which the magnitude of the closed loop transfer function has a value of 0.707, i.e., - 3 db bandwidth. The transformation of the specification into the desired constraint equation can be accomplished by following the usual procedures:

A. Find the closed loop transfer function and set s = jw.

B. Find the magnitude of the resulting expression and equate it to 0.707.

C. Substitute $w = w_b$ and simplify.

Example 2.5

For the system shown in figure 2-3, derive a relation in terms of K and K_t which assures the system has a bandwidth of 10 radians/second.

Following the prescribed procedure:

$$\frac{C(s)}{R(s)} = \frac{K}{s^2 + 5s + 5KK_T + K}$$

$$\left|\frac{C(j\omega)}{R(j\omega)}\right|^2 = \frac{K^2}{(K-\omega^2)^2 + \omega^2(5+KK_T)^2}$$

For $w = w_b = 10$, $\left|\frac{C(j\omega)}{R(j\omega)}\right|^2 = 1/2$

$$\therefore \quad K^2 = 10^4 - 200K + 100(5+KK_T)^2$$

The constraint equation obtained in the preceeding example for a second order system is typical of the expressions which are developed from bandwidth specifications. Since the expressions increase in complexity for higher ordered systems or for systems having closed loop zeroes, most design problems involving bandwidth specifications will usually require the assistance of a computer or a graphical technique. (An excellent graphical procedure based on Mitrovic's Method is given in (4).) If such aids are not available, a convenient, and relatively fast, alternative solution is to initially ignore the bandwidth specification and design the system to meet the remaining specifications. The results can then be inserted into the system bandwidth equation to determine if they also satisfy the bandwidth requirement. If the specification is not satisfied, the magnitude of the deficiency may suggest a more reasonable structure for the next attempt.

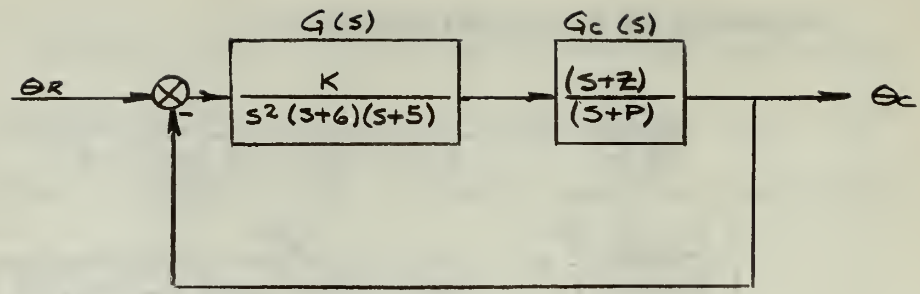

FIGURE 2.1

BLOCK DIAGRAM FOR EXAMPLE 2.3

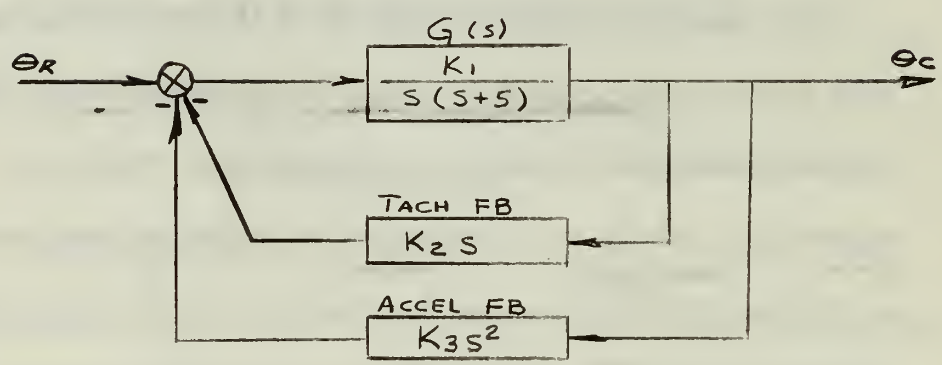

FIGURE 2.2

BLOCK DIAGRAM FOR EXAMPLE 2.4

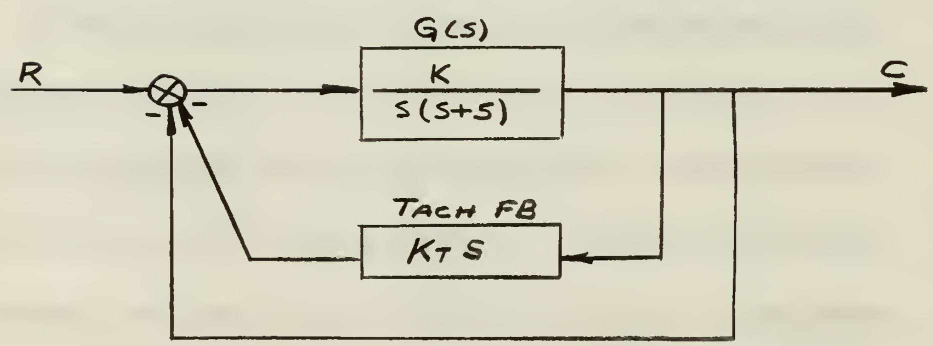

FIGURE 2.3

BLOCK DIAGRAM FOR EXAMPLE 2.5

3. Application of the Algebraic Method.

As stated previously, the basis of the proposed design method is the transformation of N specifications into a set of N simultaneous constraint equations that are explicit functions of the system parameters. Since the specifications are independent, a specific design will generally require that the system have N parameters. The insertion of sufficient parameters, usually compensating elements, is an integral part of the design procedure.

It should be noted that the mere selection of sufficient parameters does not guarantee a satisfactory solution. A converse of this statement is also true. An unsatisfactory solution does not imply that a solution does not exist; one may possibly be obtained by employing a different set of parameters. These statements suggest that there is some trial and error associated with the procedure. This is correct. However, the amount of trial and error required can be minimized through experience and application of control theory fundamentals. In this respect, root locus sketches are particularly valuable in obtaining clues to the nature of the required parameters.

The algebraic method has the inherent capability to handle any number of specifications or parameters. Therefore a parametric solution can generally be obtained by simply inserting more parameters into the system than there are design specifications. If M parameters are selected for use in the design of a system subject to N specifications, for $M > N$, the results will be parametric and expressable in terms of any $M - N$ parameters.

Example 3.1

Given $G(s) = K/s(s + 4)$, design* a system which will produce a pair of complex poles defined by $\zeta_o = 1/2$ and $w_o = 4$ with $K_v = 4$.

It will be stated without proof that $K = 16$ satisfies all specifications.

This seemingly trivial example has major significance. Specifically, one parameter successfully satisfies three specifications which is a direct contradiction of the previous discussion. It would appear that adherence to the general rule of N parameters for N specifications could possibly result in the incorporation of redundant material into the system. This does not seem likely because of the following observation:

> In all cases studied, parameters which did not assist in the solution tended to be eliminated.

Examples 3.2 and 3.3 will be used as illustrations of this observation.

Example 3.2

Insert cascade compensation, $G_c(s) = (s+Z)/(s+P)$, into the system given in example 3.1 and redesign the system to

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*Note: All designs will employ unity feedback between output and input (in addition to any other specified feedback).

meet the same specifications.

$$G_{EQ}(s) = \frac{KZ + Ks}{s^3 + s^2(4+P) + 4Ps}$$

Applying equation (2-10):

$$K_v = \frac{KZ}{4P} \qquad (1)$$

System CE:

$$s^3 + s^2(4+P) + s(4P+K) + KZ = 0$$

Mitrovic equations: $4(4+P) = 4P + K$ (2)

$$KZ = 16(4+P) - 64 \qquad (3)$$

A simultaneous solution of equations (1), (2) and (3) gives:

K = 16 P = Z

Since the pole and zero of the filter are coincident, the filter performs no function and can therefore be eliminated.

Example 3.3

For G(s) = 50/s(s + 5), it is possible to locate poles at $-5 \pm j\,5$ ($\zeta_o = 0.707$ and $w_o = 5(2)^{1/2}$) by using either tachometer feedback ($K_t = 0.1$) or a cascade compensator (Z = 5, P = 10). In each case, $K_v = 5$. Design a system using both tachometer feedback and a series compensator to satisfy the same specifications.

For combined tachometer feedback and cascade compensation:

$$G_{EQ}(s) = \frac{50(s+Z)}{s^3 + s^2(P+5+50K_T) + s(5P+50K_TZ)}$$

Applying equation (2-10): $K_v = 50Z/(5P + 50K_TZ)$ (1)

System CE:

$$s^3 + s^2(P+5+50K_T) + s(5P+50K_T z+50) + 50z = 0$$

Mitrovic equations: $50z = 50(P+5+50K_T) - 500$ (2)

$$5P + 50K_T z + 50 = 10(P+5+50K_T) - 50 \qquad (3)$$

Simplifying: $Z = 5$ (4)

$$P = 10(1 - 5K_t) \qquad (5)$$

The result obtained in this example is doubly significant. First of all, it indicates that the procedure is capable of detecting conditional as well as complete redundancy. In example 3.2, the filter was redundant for all values of the compensator pole and zero. In this example, however, equation (5) shows that the filter is redundant only when $K_t = 0.1$. Similarly, the tachometer may be discarded when P = 10. The second point of interest is the fact that a unique solution may not exist. In this case, any positive values of P and K_t which satisfy equation (5) also satisfy the specifications. Which solution is best? At this point, it is a matter of conjecture. However, this example will be rediscussed in Section 4 after a means for determining the best design has been developed.

Example 3.4

Given the forward loop transfer function, $G(s) = \frac{K}{s(s+1/2)(s+7)}$, design series compensation to give roots with $\zeta_o = 1/2$ and $w_o = 6$. The minimum acceptable value of K_v is 4.15.

For cascade compensation:

$$G_{EQ}(s) = \frac{K(s+z)}{s^4 + 7.5Ps^3 + s^2(7.5P+3.5) + 3.5Ps}$$

Applying equation (2-10): $K_v = 4.15 = \dfrac{Kz}{3.5P}$ (1)

System CE:

$$s^4 + 7.5Ps^3 + s^2(7.5P+3.5) + s(3.5P+K) + Kz = 0$$

Mitrovic equations: $Kz = 54P - 1495$ (2)

$K = 41.5P - 195$ (3)

Solving equations (1), (2) and (3) gives: K = 1475

P = 37.9

Z = 0.373

Example 3.5

The bandwidth for the system in example 3.4 is about 6.5 radians/second. Redesign the system so that the bandwidth is reduced to 4 radians/second.

The introduction of another specification requires the insertion of another parameter. Use tachometer feedback around the original plant and series compensator. Then

$$G_{eq}(s) = \frac{K(s+z)}{s^4 + s^3(7.5+P) + s^2(3.5+7.5P+KK_T) + s(3.5P+KK_Tz)}$$

Applying equation (2-10): $K_v = 4.15 = \dfrac{Kz}{3.5P+KK_Tz}$ (1)

System CE:

$$s^4 + s^3(7.5+P) + s^2(3.5+7.5P+KK_T) + s(3.5P+KK_Tz+K) + Kz = 0$$

Mitrovic equations: $Kz = 6^2(3.5+7.5P+KK_T) - 6^3(P+7.5)$ (2)

$3.5P + KK_Tz = 6(3.5+7.5P+KK_T) - 6^3$ (3)

System bandwidth equation for $w_b = 4$:

$$1/2 = \frac{(KZ)^2 + 16K^2}{[KZ - 16(3.5 + 7.5P + KK_T) - 256]^2 + 16[3.5 + KK_TZ + K - 16(P + 7.5)]^2} \quad (4)$$

Solving equations (1) through (4):

$K = 252 \qquad P = 5.93$

$K_t = 0.16 \qquad Z = 1.015$

4. Sensitivity as a Specification.

Attention is now directed to the question posed during the discussion of example 3.3. If more than one design satisfies the specifications, which is the best design? The fact that a specific solution was not obtained indicates the need for another equation which, for the system being developed, implies the need for another specification. Since the design specifications have already been incorporated into the solution, it follows that the additional specification should be general in nature and applicable to all systems.

One choice for a general control system specification is sensitivity which will be defined as

$$S_Q^P \equiv \frac{d(\ln P)}{d(\ln Q)} = \frac{dP/P}{dQ/Q} = \frac{Q}{P}\left(\frac{dP}{dQ}\right) \tag{4-1}$$

where S_Q^P is read as "the sensitivity of a function P with respect to a parameter Q." (This definition is preferred by the author because it imparts more physical meaning than most other definitions of sensitivity found in the literature, i.e., it states that sensitivity is essentially the ratio of the percentage change in the function to the percentage change in the parameter.) The reason for this choice is the fact that a sensitivity specification will force the final design to be realistic. Specifically, it requires the consideration of changes in the parameters of the physical system which will result because of tolerances, aging and environment. On this basis, the best design is the one which not only satisfies the design specifications, but is also able to maintain the desired performance in spite of undesired parameter variations.

Root Sensitivity

Use of equation (4-1) requires definition of the function P. Since sensitivity is being introduced as a means of insuring compliance with the design specifications, P should ideally be a function of all the design specifications. However, in this presentation, only the root specification, i.e., ζ_o and w_o, will be considered. Hence, the discussion will refer to the root sensitivity of the system. (Hereafter, root sensitivity and pole sensitivity will be used interchangeably.)

Restricting attention to root sensitivity does not completely preclude the determination or discussion of a best design. In fact, there are two arguments which partially justify (and might even suggest) this simplification. One argument is that root specifications are more stringent than either the bandwidth or error coefficient specifications. The latter two are frequently specified in terms of inequalities, i.e., greater or smaller than a certain minimum or maximum value. Therefore, some tolerance can be built into the design by working on the "safe side" of the specifications. Another argument is based on the root-coefficient relations of the system characteristic equation. If the mobility of two of the systems' closed loop poles are restricted by means of a sensitivity constraint, then the mobility of the remaining closed loop poles is also restricted. This statement stems from the fact that the specified poles have corresponding roots which must be factors of the characteristic equation. Now, in order for these roots to be perfect factors of the characteristic equation, the coefficients of the characteristic equation must have remained reasonably constant for small parameter changes. This implies that the remaining roots, and hence the remaining poles, also remained reasonably constant. Since both bandwidth and the error coefficients are partially dependent (and, in fact, totally dependent for low order systems without zeroes) on the characteristic equation, it follows that their sensitivity functions bear an approximate relation to the root sensitivity functions.

Root Sensitivity Equations

Although the practical importance of sensitivity has long been recognized, it has only been in recent years that effort was expended in the development of sensitivity as a useful tool. The work of Kokotović and Šiljak, (2), is considered to be a major advancement towards this goal. In essence, they used the definition of sensitivity, equation (4-1), in conjunction with the generalized Mitrovic equations, equations (2-6) and (2-7), to obtain algebraic expressions for (what the author considers to be) the components of root sensitivity. Their results are not only particularly suited for use in the design of control systems based on a specified pole pair, but also simplify sensitivity computations because they permit the exclusive use of real variables in calculating the sensitivity of a function containing complex variables.

Since the Kokotović and Šiljak sensitivity equations will be used in subsequent examples, a brief review of their development will now be presented. Differentiating the generalized Mitrovic equations, equations (2-6) and (2-7), with respect to a parameter Q gives

$$\frac{1}{\omega_n}\frac{\partial \omega_n}{\partial Q}\sum_{K=0}^{n} K a_K \omega_n^K \phi_{K-1}(\zeta) + \frac{\partial \zeta}{\partial Q}\sum_{K=0}^{n} a_K \omega_n^K \phi'_{K-1}(\zeta) + \sum_{K=0}^{n}\frac{\partial a_K}{\partial Q}\omega_n^K \phi_{K-1}(\zeta) = 0 \tag{4-2}$$

$$\frac{1}{\omega_n}\frac{\partial \omega_n}{\partial Q}\sum_{K=0}^{n} K a_K \omega_n^K \phi_K(\zeta) + \frac{\partial \zeta}{\partial Q}\sum_{K=0}^{n} a_K \omega_n^K \phi'_K(\zeta) + \sum_{K=0}^{n}\frac{\partial a_K}{\partial Q}\omega_n^K \phi_K(\zeta) = 0 \tag{4-3}$$

where $\phi'_k(\zeta)$ and $\phi'_{k-1}(\zeta)$ are the derivatives of $\phi_k(\zeta)$ and $\phi_{k-1}(\zeta)$ respectively and are related by the general equation

$$\phi'_K(\zeta) = -2\left[\zeta\,\frac{K-1}{K-2}\,\phi'_{K-1}(\zeta) + \frac{K}{K-2}\,\phi'_{K-2}(\zeta)\right] \tag{4-4}$$

Numerical values for ϕ'_k for various values of ζ are given in Appendix II. Manipulation of equations (4-2) and (4-3) and application of equation (4-1) gives

$$A_1 S_Q^{\omega_n} + \zeta B_1 S_Q^{\zeta} = -QC_1 \tag{4-5}$$

$$A_2 S_Q^{\omega_n} + \zeta B_2 S_Q^{\zeta} = -QC_2 \tag{4-6}$$

where

$$A_1 = \sum_0^n K a_K \omega_n^K \phi_{K-1}(\zeta) \qquad A_2 = \sum_0^n K a_K \omega_n^K \phi_K(\zeta) \tag{4-7}$$

$$B_1 = \sum_0^n a_K \omega_n^K \phi'_{K-1}(\zeta) \qquad B_2 = \sum_0^n a_K \omega_n^K \phi'_K(\zeta) \tag{4-8}$$

$$C_1 = \sum_0^n \frac{\partial a_K}{\partial Q} \omega_n^K \phi_{K-1}(\zeta) \qquad C_2 = \sum_0^n \frac{\partial a_K}{\partial Q} \omega_n^K \phi_K(\zeta) \tag{4-9}$$

Solving equations (4-5) and (4-6) for S_Q^{ζ} and $S_Q^{w_n}$ gives

$$S_Q^{\zeta} = -\frac{Q}{\zeta} \frac{D_Q^{\zeta}}{D} \tag{4-10}$$

$$S_Q^{\omega_n} = -Q \frac{D_Q^{\omega_n}}{D} \tag{4-11}$$

where

$$D = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \tag{4-12}$$

$$D_Q^{\omega_n} = \begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix} \tag{4-13}$$

$$D_Q^{\zeta} = \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} \tag{4-14}$$

Equations (4-10) and (4-11) are the desired sensitivity design equations. (Hereafter, S_Q^{ζ} and $S_Q^{w_n}$ will be referred to as the "component sensitivity functions.") At this time, their true simplicity is masked by their dependence on the solution of a number of auxilliary equations. Suffice to state that the solutions of the auxilliary equations become exceedingly simple when the location of a pole pair is specified (ζ_o and w_o) and the coefficients of the characteristic equation are known (a_K^o). This will be clearly demonstrated in subsequent examples.

Intrepretation of Sensitivity Values

Equations (4-10) and (4-11) may be used in two ways. Substitution of a given set of parameter values into the right hand side of the equations gives a definite sensitivity value for a specific set of parameter values. Alternatively, the assignment of a sensitivity value to the component sensitivity functions give the parameter relations which must be satisfied in order to obtain a system of the specified sensitivity.

In either case, the final design will be dependent upon the accepted/assigned value to sensitivity. It follows that some guidelines for reasonable sensitivity values must be established. Since the literature is lacking in this area, the following guides (which are consistent with the original interpretation of equation (4-1)) are proposed by the author:

A. Signs (+ or -) of sensitivity values indicate sense only. Therefore, the minimum value of sensitivity is zero, implying that a function is totally insensitive to variations of a parameter.

B. $\left|S_Q^P\right| = 1$ indicates a linear variation in P due to changes in Q.i.e., a 5% change in Q results in a 5% change in P. This is considered to be the maximum permissible value of sensitivity; if possible, systems should have sensitivity values much less than one.

C. $\left|S_Q^P\right| > 1$ indicates amplification of the parameter variation. This is generally considered undesireable for a control system (but may have application in some detection schemes).

D. As indicated in A above, sensitivity values have vector properties. Therefore, the total root sensitivity, S_Q^{RT}, is appropriately expressed as

$$(S_Q^{RT})^2 = (S_Q^{\zeta})^2 + (S_Q^{\omega_n})^2 \qquad (4\text{-}15)$$

Sensitivity as a Specification

The concept of sensitivity has been developed to the extent that it can now be used as an additional system constraint subject to analysis by the algebraic design method. The remainder of this section will be devoted to the design and discussion of specific systems which must satisfy sensitivity, as well as other, specifications.

Example 4.1

Reconsider example 3.3 in view of a specification requiring minimum sensitivity of the specified poles with respect to variations in system gain, K.

Considering the system gain, K, as a parameter, the system characteristic equation is

$$s^3 + s^2(p+5+KK_T) + s(5p+KK_T z+K) + Kz = 0 \qquad (1)$$

The sensitivity equations will now be devloped in detail to demonstrate the solution of the auxilliary equations. Applying equations (4-7) and (4-8) gives

$$\left.\begin{aligned} A_1 &= 1500\,a_3 - 100\,a_2 \\ A_2 &= 100\sqrt{2}\,a_2 - 5\sqrt{2}\,a_1 - 750\sqrt{2}\,a_3 \\ B_1 &= 500\sqrt{2}\,a_3 \\ B_2 &= 100\,a_2 - 2000 \end{aligned}\right\} \quad (2)$$

after substituting $w_n = w_o = 5(2)^{1/2}$ and values for $\phi_k(0.707)$ and $\phi'_{k-1}(0.707)$ from Appendices I and II. Although specific values of P and K_t are unknown, inspection of equation (1) in conjunction with the results of example 3.3 show that the coefficients of the characteristic equation must have the following values for all solutions

$$a_0^\circ = 250 \qquad a_1^\circ = 100 \qquad a_2^\circ = 15 \qquad a_3^\circ = 1$$

which, when substituted into equation (2), gives

$$A_1^\circ = 0 \qquad A_2^\circ = 250\sqrt{2} \qquad B_1^\circ = 500\sqrt{2} \qquad B_2^\circ = -500 \qquad (3)$$

Taking the partial of each coefficient in equation (1) with respect to K gives

$$\frac{\partial a_0}{\partial K} = 0 \qquad \frac{\partial a_1}{\partial K} = 1 + K_T Z \qquad \frac{\partial a_2}{\partial K} = K_T \qquad \frac{\partial a_3}{\partial K} = 0 \qquad (4)$$

Applying equations (4-9) and substituting, as above, for w_n and $\phi_k(0.707)$ gives

$$C_1 = Z - 50K_T \qquad C_2 = 50\sqrt{2}\,K_T - 5\sqrt{2}(K_T Z + 1)$$

which, for Z = 5 as obtained in example 3.3, reduces to

$$C_1 = 5(1 - 10K_T) \qquad C_2 = 5\sqrt{2}(5K_T - 1) \qquad (5)$$

Applying equation (4-12) and substituting equation (3) gives

$$D = -250 \times 10^3 \qquad (6)$$

Applying equation (4-13) and substituting from equations (3) and (5) gives

$$D_K^{\omega_n} = 2500 \tag{7}$$

Applying equation (4-14) and substituting from equations (3) and (5) gives

$$D_K^{\zeta} = -1250\sqrt{2}\,(1 - 10K_T) \tag{8}$$

Substituting Q = K = 50, $\zeta = \zeta_o = 0.707$ and equations (6), (7) and (8) into equations (4-10) and (4-11) gives

$$S_K^{\zeta} = 1/2\,(10K_T - 1) \tag{9}$$

$$S_K^{\omega_n} = 1/2 \tag{10}$$

Equations (9) and (10) are the desired design equations. Since only one additional constraint equation was needed, they would normally be squared and added to obtain an expression for the total root sensitivity. In this case, however, $S_K^{w_n}$ is observed to be a constant. Therefore, the minimum pole sensitivity with respect to K can be found by setting equation (10) equal to zero. This gives $K_t = 0.1$ which corresponds to the solution of tachometer feedback only.

For $K_t = 0.1$, $S_k^{w_n} = 1/2$ and $S_k = 0$. Therefore, the analysis predicts that a ten per cent change in K should produce a five per cent change in w_n while ζ remains constant. These predictions can be mathematically verified. The system characteristic equation for the specified value of K_t is

$$s^2 + s(KK_T + 5) + K = 0$$

For K = 50:	$w_o = 7.07$	$\zeta_o = 0.707$
For K = 55:	$w_n = 7.42$	$\zeta = 0.708$

Therefore $\frac{\Delta\omega_n}{\omega_0} = 4.95\%$ $\frac{\Delta\zeta}{\zeta_0} = 0.14\%$

Parameter Selection

The preceeding example raises several questions which have yet to be discussed. The first deals with the selection of a parameter on which to base the sensitivity analysis. One choice might be the parameter which is considered to have the widest variations during expected conditions of operation, e.g., the gain of an electronic amplifier. Another choice might be similar to the first but restricted to plant parameters. This assumes the designer is able to select and/or control the sensitivity of the required compensating elements. There is no guarantee that either is correct. It may be possible that the system is naturally insensitive to the parameter likely to have the widest variations in value while it could be highly sensitive to a parameter that is expected to have only minute variations in value. It is also possible that, by minimizing the sensitivity due to one parameter, the effect of another is amplified. The answer, then, requires investigation of all possible parameters. It is believed that a thorough investigation of a particular type of system could lead to the establishment of general guides for the selection of the appropriate parameter(s) in other similar systems. Since such an investigation is not consistent with the purpose of this paper, the author will be content to arbitrarily select any parameter which will facilitate demonstration of the algebraic method.

It should also be noted that there are no restrictions as to the number of parameters which may be similtaneously considered. If the sensitivity of pole P with respect to parameters Q_1, Q_2 and Q_3 is desired, the component sensitivity functions for each parameter may be individually calculated. The total component sensitivity function due to all three parameters is the algebraic sum of the individual components, e.g.,

$$S^{\omega_n}_{Q_1 Q_2 Q_3} = S^{\omega_n}_{Q_1} + S^{\omega_n}_{Q_2} + S^{\omega_n}_{Q_3}$$

. Since the sense of each component must be considered, there is a possibility that individual components may cancel. Another fortunate circumstance is that the computations for a situation considering the root sensitivity with respect to N parameters do not increase N-fold. This is because equations (4-7), (4-8) and (4-12), i.e., A_1, A_2, B_1, B_2 and D, remain the same for all parameters.

Minimum Sensitivity

Another point of interest in example 4.1 is the ease with which the condition of minimum sensitivity was obtained. This is not a unique situation because it was possible to obtain results by inspection in other systems studied also. However, it seems quite likely that for many systems the parameter relations for minimum sensitivity must be obtained by setting the derivative of the sensitivity function equal to zero. (Note the problem if the sensitivity with respect to more than one parameter is being considered.) In such cases, care must be taken to insure that the function to be differentiated is in a form which will indicate minima near zero. This is necessary because minimum sensitivity was defined to be zero. The above stipulation can be conveniently accomplished by considering either the absolute value or the square of the component sensitivity function, i.e., $|S_Q^{\omega_n}|$ or $(S_Q^{\omega_n})^2$ vice $S_Q^{\omega_n}$. Use of the total sensitivity function, equation (4-15), is also appropriate.

Discussions thus far have repeatedly used the term "minimum sensitivity." This does not necessarily mean that minimum sensitivity must be attained by, or even an objective of, every design. The specified sensitivity value, whether it be zero or two, should be consistent with the other specifications. For example, if the pole locations are only required to be accurate within five per cent for satisfactory system response and the parameter of interest is also expected to have five per cent variations, a sensitivity value of one is sufficient. Of course, if analysis shows that a lower value is attainable, the design should be based on the lower value provided it can be achieved without additional cost or degradation of some other aspect of system performance.

The nature of the sensitivity function in the vicinity of the design value is also of concern. If the design value is associated with a point on the sensitivity function curve where the curve is very steep or irregular, the system is likely to be more sensitive than an alternative design based on a (reasonably) higher value which corresponds to a smooth portion of the sensitivity function curve.

Example 4.2

A portion of a control system has the forward transfer function, $G(s) = K/s(s + 2)(s + 4)$. Dominant closed loop poles with $\zeta_o = 1/2$ and $w_o = 5$ are required for the desired response. Overshoot is of prime importance; therefore, ζ_o should be maintained within 1% accuracy for the anticipated gain variations of 5%. Velocity and acceleration signals are required elsewhere in the system and are therefore available for use as feedback signals.

The problem as stated gives 3 specifications (pole pair and S_k^{ζ}) and 3 parameters. (Note that an additional specification concerning S_K^{Wn} would normally require the insertion of an additional parameter whereas a specification regarding the total pole sensitivity would not. Also note that the specifications permit w_o to vary, if necessary, in order to maintain ζ_o approximately constant.)

System CE: $s^3 + s^2(6 + KK_a) + s(8 + KK_T) + K = 0$

Mitrovic equations: $$K = 25 + 25\,KK_a \qquad (1)$$

$$KK_T = 22 + 5\,KK_a \qquad (2)$$

Applying equations (4-7) thru (4-9):

$$\left.\begin{array}{ll} A_1 = 375 - 50a_2 & A_2 = 50a_2 - 5a_1 \\ B_1 = 250 & B_2 = 50a_2 - 500 \\ C_1 = 1 - 25KK_a & C_2 = 5(5K_a - K_T) \end{array}\right\} \qquad (3)$$

Unlike example 4.1, the coefficients of the characteristic equation do not have fixed numerical values. Therefore, the solution of the determinants for D and D_k, equations (4-12) and (4-14), will be hampered by the presence of all 3 parameters. At this point it will be convenient to solve equations (1) and (2) for K_a and K_t. Then equation (3) can be expressed in terms of K only.

$$K_a = \frac{K - 25}{25K} \qquad K_T = \frac{K + 85}{5K} \qquad (4)$$

Therefore

$$\left.\begin{array}{ll} A_1 = 125 - 2K & A_2 = K + 125 \\ B_1 = 250 & B_2 = 2K - 250 \\ C_1 = 25/K & C_2 = -110/K \end{array}\right\} \qquad (5)$$

Using equation (5) to solve for D and D_k^{ξ}:

$$D = -4K^2 + 500K - 250^2 \qquad D_K^{\xi} = \frac{1}{K}(195K - 135 \times 125) \qquad (6)$$

Using equation (6) to solve equation (4-13:

$$S_K^{\xi} = \frac{97.5\,(K - 86.5)}{K^2 - 125K + 125^2} \qquad (7)$$

Inspection of equation (7) shows that $S_k^{\xi} = 0$ for K = 86.5. Since the only requirement is that $S_k^f \leq 0.2$, this gain setting should be considered as a possible design value. For K = 86.5, $K_t = 0.397$ and $K_a = 0.0284$. For these values, the third closed loop pole is located at $\sigma = -3.4$. Therefore, the complex poles are not completely dominant. In addition, figure 4-1 shows the sensitivity function is very steep at the point of minimum sensitivity. Again using figure 4-1, the system will be designed on the basis of K = 600. The other design values are: $K_t = 0.238$ and $K_a = 0.0384$. The complex poles are assured of being dominant because the third pole is now located at $\sigma = -24$. It is stated without proof that $S_k^{w_n}$ also approaches zero as K becomes very large. It is also less sensitive than S_k^f, having a value < 0.05 at the design gain setting.

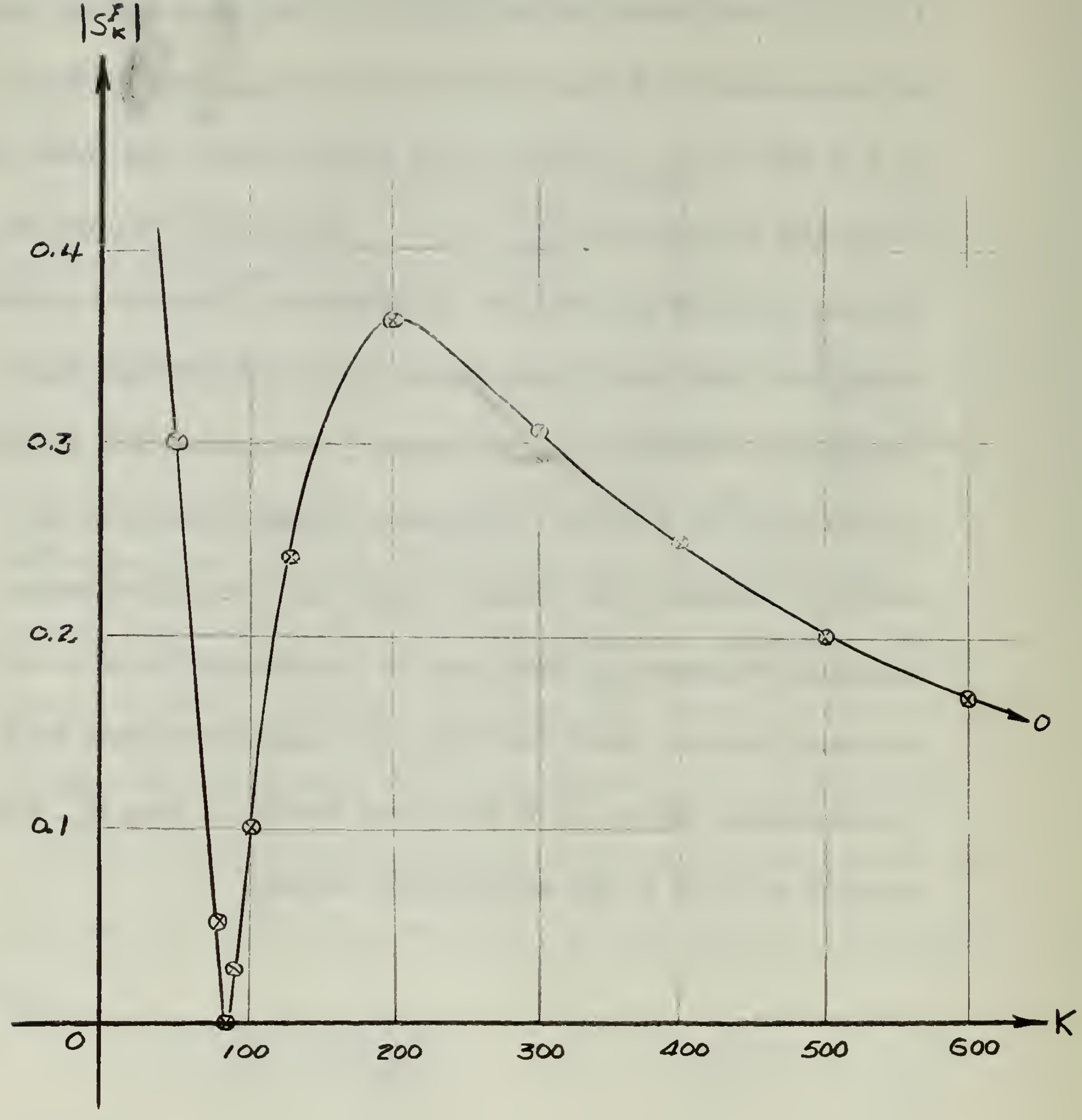

FIGURE 4-1

SENSITIVITY (S_K^F) VS GAIN (K)

FOR EXAMPLE 4-2

5. Conclusions

The purpose of this paper was to develop a design technique based on the premise that design can be accomplished by solving a set of algebraic simultaneous equations derived from the system design specifications. In addition to demonstrating the feasibility of this concept, the examples in the preceeding sections also showed that the method developed has certain desirable characteristics. Specifically, the algebraic method is analytically simple, capable of handling any number of parameters, rapid, specific and accurate.

The method is in no way restricted to the exclusive use of the design specifications considered in this paper. The specifications considered were simply selected to enhance the presentation. In general, a system can be designed to satisfy any set of independent specifications providing the specifications can be expressed as algebraic equations. It is obvious, of course, that the solution of the resulting equations becomes more difficult as the number of specifications increase. For this reason, it is suggested that analysis be normally limited to specifications which are considered to be prime, i.e., those which are exactly specified as opposed to others stated in terms of inequalities or approximations. The results of this analysis can then be used to determine if the secondary specifications are also satisfied.

The inclusion of sensitivity equations into the set of system constraint equations was considered especially significant. Here again, the use of sensitivity equations is in no way dependent upon the assumptions made in Section 4 concerning the relation of root sensitivity to system sensitivity, i.e., use of root sensitivity as a performance index. This was logically demonstrated in example 4.2 where a root sensitivity component was of particular concern while the system sensitivity had no restrictions whatsoever. Although only root sensitivity was discussed, there is no apparent reason why similar restrictions cannot be imposed on other design specifications.

As previously indicated in Section 4, there is a definite need for the study of specification/parameter sensitivity relationships before sensitivity can become a useful design tool. At the present, the sensitivity effects of all possible parameters must be evaluated with respect to each specification and to each other before a meaningful design can be achieved. Analysis of even a simple system on this basis would require a prohibitive amount of computation because of the large possible number of parameter/specification combinations available.

"Seek Simplicity and Distrust It"

Alfred North Whitehead

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APPENDIX I

$\phi_k(\mathcal{F})$ FUNCTIONS

$\mathcal{F}$	ϕ_{-1}	ϕ_0	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7
0.00	1	0	-1	0.0	1.00	0.000	-1.0000	0.00000	1.000000
0.05	1	0	-1	0.1	0.99	-0.199	-0.9701	0.29601	0.940499
0.10	1	0	-1	0.2	0.96	-0.392	-0.8816	0.56832	0.767936
0.15	1	0	-1	0.3	0.91	-0.573	-0.7381	0.79443	0.499771
0.20	1	0	-1	0.4	0.84	-0.736	-0.5456	0.95424	0.163904
0.25	1	0	-1	0.5	0.75	-0.875	-0.3125	1.03125	-0.203125
0.30	1	0	-1	0.6	0.64	-0.984	-0.0496	1.01376	-0.558656
0.35	1	0	-1	0.7	0.51	-1.057	0.2299	0.89607	-0.857149
0.40	1	0	-1	0.8	0.36	-1.088	0.5104	0.67968	-1.054144
0.45	1	0	-1	0.9	0.19	-1.071	0.7739	0.37449	-1.110941
0.50	1	0	-1	1.0	0.00	-1.000	1.0000	0.00000	-1.000000
0.55	1	0	-1	1.1	-0.21	-0.869	1.1659	-0.41349	-0.711061
0.60	1	0	-1	1.2	-0.44	-0.672	1.2464	-0.82368	-0.257984
0.65	1	0	-1	1.3	-0.69	-0.403	1.2139	-1.17507	0.313691
0.70	1	0	-1	1.4	-0.96	-0.056	1.0384	-1.39776	0.918464
0.75	1	0	-1	1.5	-1.25	0.375	0.6875	-1.40625	1.421875
0.80	1	0	-1	1.6	-1.56	0.896	0.1264	-1.09824	1.630784
0.85	1	0	-1	1.7	-1.89	1.513	-0.6821	-0.35343	1.282931
0.90	1	0	-1	1.8	-2.24	2.232	-1.7776	0.96768	0.035776
0.95	1	0	-1	1.9	-2.61	3.059	-3.2021	3.02499	-2.545381
1.00	1	0	-1	2.0	-3.00	4.000	-5.0000	6.00000	-7.000000

APPENDIX II

$\phi'_k(\mathcal{F})$ FUNCTIONS

$\mathcal{F}$	ϕ'_2	ϕ'_3	ϕ'_4	ϕ'_5	ϕ'_6	ϕ'_7	ϕ'_8	ϕ'_9	ϕ'_{10}
0.00	2	0.0	-4.00	0.0000	8.000	0.0000	-10.6666	0.0000	13.3333
0.05	2	-0.4	-3.94	1.192	5.761	-2.1401	-6.1401	3.4911	7.6960
0.10	2	-0.8	-3.76	2.336	5.056	-4.4838	-5.6941	7.0666	5.5327
0.15	2	-1.2	-3.46	3.921	3.921	-6.1491	-3.0758	8.9605	0.8205
0.20	2	-1.6	-3.04	4.288	2.516	-7.1629	0.1213	9.1534	-4.2706
0.25	2	-2.0	-2.50	5.000	0.625	-7.3750	3.4687	7.5000	-8.5546
0.30	2	-2.4	-1.84	5.472	-1.344	-6.6931	6.6931	4.1639	-10.9071
0.35	2	-2.8	-1.06	5.656	-3.359	-5.0968	8.6410	-0.3598	-10.5179
0.40	2	-3.2	-0.16	5.504	-5.264	-2.6521	9.4939	-5.2703	-7.1241
0.45	2	-3.6	0.86	4.968	-6.879	0.4741	8.6742	-9.5316	-1.1920
0.50	2	-4.0	2.00	4.000	-8.000	4.0000	6.0000	-12.0000	6.0000
0.55	2	-4.4	3.26	2.555	-8.403	7.5151	1.5597	-11.6230	12.4338
0.60	2	-4.8	4.64	0.576	-7.824	10.4601	-4.2121	-8.6349	16.9222
0.65	2	-5.2	6.14	-1.976	-6.010	12.1420	-10.4020	-0.1567	13.2317
0.70	2	-5.6	7.76	-5.152	-2.624	11.6211	-15.4491	9.7771	3.9124
0.75	2	-6.0	9.50	-9.000	2.625	7.8750	-17.2812	19.4999	-11.3048
0.80	2	-6.4	11.36	-13.568	10.096	-0.3891	-12.7350	23.7871	-26.8980
0.85	2	-6.8	13.34	-18.904	20.161	-14.6628	2.1998	14.5782	-30.6305
0.90	2	-7.2	15.44	-25.056	33.216	-36.6681	32.7150	-20.1547	-0.0804
0.95	2	-7.6	17.66	-32.072	49.681	-68.3719	107.2698	-145.0219	175.8970
1.00	2	-8.0	20.00	-40.000	70.000	-112.0000	168.0000	-230.0000	300.0000

$\phi'_0 = \phi'_1 = 0;\ \phi'_{-k} = -\phi'_k$

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13. ABSTRACT

A method is presented for the analytic design of linear control systems. Design is accomplished by the solution of an algebraic set of simultaneous equations derived from the design specifications. Specifications considered include the system damping ratio, undamped natural frequency, bandwidth, steady state error coefficients and root sensitivity.

The method is shown to be fast, accurate, specific and capable of working with any number of parameters. In addition, the analytic approach minimizes dependence on graphical techniques and trial and error solutions.

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